Психология. Журнал Высшей школы экономики,
2014. Т. 11. № 3. С. 55–78.

PERCEPTUAL ACTION OF NOVICES AND EXPERTS IN OPERATING VISUAL REPRESENTATIONS OF A MATHEMATICAL CONCEPT

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Research is conducted with the support of Russian Foundation for Humanities, project № 12-36-01408.

Abstract
In this paper we explore the perceptual actions that allow one to perceive pictures as representing mathematical concepts. The research is based on the cultural–historical approach. Following V.V. Davydov's ideas on theoretical and, particularly, mathematical thinking, we consider a mathematical concept as being based on a historically determined method of action. Using the eye-tracking system we analyzed the difference between school students, university students, and expert mathematicians in their perception of special pictures (so called "external visual representations of the theoretical concept") when performing a set of tasks, namely choosing a point with given coordinates from a set of points. A standard expert-novice research analysis of dwell time in relevant and irrelevant areas of interest was used. We also compared the gaze paths, the number of visual fixations, and the time each group required to perform the tasks. The directions of the saccades were also analyzed, and our data showed that the vertical and horizontal saccades along the axes prevailed over saccades along other directions, a fact that may be considered as a trace of the "Cartesian plane" concept. The data showed that experts performed the tasks faster and with fewer fixations and they also were able to use additional knowledge flexibly in organizing their perceptive actions. Our results show the fundamental interlacing of conceptual structures and visual processes, in which the latter are organized in accordance with prior knowledge. The specificity of the experts' Cartesian plane perception corresponds to the late stages of the historical development of this concept. We consider this fact as an empirical confirmation of the relevance of the term "theoretical perception".

Keywords: logical-historical analysis, perception, perceptual actions, visual representation, mathematical concepts, Cartesian coordinates, eye-tracking, novices and experts, psychology of mathematics education.

Introduction

The subject of our research is the system of knowledge and methods of action that is formed in students of various levels of mathematics competence and specialists-mathematicians around the mathematical concept of 'Cartesian coordinates'. We view this concept as an example of a mathematical concept in general, and think that its study allows us to make more or less general conclusions about the functioning of mathematical concepts in general. The choice of this concept is determined by its specific features which make it suitable for studying using the contemporary method of eye-tracking recording.

Our approach may be associated with the broad field of Cultural-Historical psychology and Activity theory, but we actively use the results of other approaches, and so we need to define comparable and translatable terminologies

The experiment will be described in further detail below, but for now we offer a short summary to introduce and elucidate what follows. To begin with, the text of the task — which was the request to find a point with definite coordinates, for example (2, 4) — was shown to a participant and then the plot appeared with coordinate axes and several points marked on it. We were interested in the participants' perceptive activity in searching for a point that met the conditions of the task.

We call such a plot, namely the Cartesian plain depicted in the task, a model of the "Cartesian coordinates" concept, which substitutes the concept in a sensually perceived form. In order to see that this model doesn't cover the full meaning of the concept, it is enough to note that the potential of the algebraic aspects of this concept (which were implied by R. Descartes and P. Fermat at the time of the concept's creation) is not realized in this model. The model provides only the possibility of finding the coordinates of the point and to find the point with the coordinates by drawing appropriate projections.

Generally, we consider the *model* of a mathematical concept as the essential features inherent in the mathematical concept, which are fixed in the material form. A model is a symbolic formation that allows action and interaction with the mathematical concept, which is accessible for manipulations through no means other than models. We call *visual* models those that represent the concept in a visually accessible spatial form. We use this term as a partial analog of the English term "visual representation", widely used in studies of mathematical education, executed in semiotic approach, where a mathematical concept is viewed as an integration of several representations. However, the term "visual representations" seems inappropriate for us (this is also noted by Presmeg (2006)), as it mixes internal representations — mental images and spatial notions — with external representations that have concrete material substrate and are objectified with the help of it.

The material object, which was initially created as a model, acquires the "modeling" or "representative" function for a subject only when the subject *correctly perceives this object in the context of those features that were initially inherent in the concept.* (Otherwise the parabola graph may become the picture of a "vase", as one of the students remarked in our interview, and doesn't bear any modeling function). Moreover, as Presmeg (2008) reasonably stresses, the ascription of a model (a sign) to a visual (iconic) or symbolic type is based only on a way of interpretation and cannot be unambiguously done in isolation from the subject, who endows the given material object with the meaning.

In recent years the importance of visual models for the acquisition of mathematical knowledge has become increasingly obvious. The general objective of this work is to study how visual models are incorporated into the mathematical knowledge of a subject and how the subject's perception of visual models is transformed due to the acquisition of new knowledge.

As V.V. Davydov wrote, the creation of a scientific model is the result of a long investigation that embodies the transformation of reality by object-related actions. This embodiment makes explicit essential features of objects which were hidden before modeling: "Models are a form of scientific abstraction of a particular kind, in which the essential relationships of an object which are delineated are reinforced in visually perceptible and represented connections and relationships of material or symbolic elements" (Davydov, 1990/1972, p. 122).

Does it mean that the natural perception of a visual mathematical model gives immediate access to mathematical knowledge, immediate access to the meaning of the presented picture? V.V. Davydov wrote that the model is "a distinctive form of connection between the sensory and the rational in cognition" (Ibid.), and that perception of a model "is inseparably related to the theoretical interpretation of its structure" (Ibid.). He followed these words with a quotation from V.A. Shtoff: "Visuality in the perception of a visual model presupposes, at the same time, significant participation by thought, the application of accumulated theoretical knowledge, accumulated experience" (Ibid.). Thus, the meaning of visual mathematical models is certainly not immediate perception possible for anyone; correct perception has to be specially taught, and that view is also articulated by contemporary researchers (Aspinwall, Shaw, & Presmeg , 1997; Duval, 2008; Radford, 2010, 2013; Presmeg, 1992).

Therefore, there is theoretical thinking within the process perception when pictures are viewed not as simple pictures but "the particular cognitive attitude to drawings and diagrams, special methods of 'reading' them" (Davydov, 1990/1972, p. 73) are realized in them. We assume that such perception may be characterized as "*theoretical*" in parallel with "theoretical form of perception" and "eye as theoretician" by L. Radford (2013, p. 62).

A perception of a model is a theoretical perception when the model is perceived in the context of its representational function towards the mathematical concept. In order to see a visual model as a representative one, one needs to uncover its essential inner relationships with the concept, which may not be visible with a "naïve" perception. This "transformation of a 'latent' property into an 'explicit' one" (Davydov, 1963, p. 140) becomes possible due to special "*theoretical*" actions, which initially occur in objective form and then intervene in the process of perception. Thus, the perception itself becomes *theoretical*, i.e. based on an action of reality transformation, disclosing initially inner "latent" properties: "In discussing action, we have in mind primarily sensory-object, cognitive action. Therefore it is still 'sensory' — and does it reveal internal connections? Yes, it is sensory, but with an important addition — an object-related action, really changing the object of study, experimenting on it. It has its own prototype in practical-object action" (Davydov, 1990/1972, p. 126). Thus the perception of a visual model in order to be cultural and adequate to the system of mathematical knowledge has to include traces of actions on which the elaboration of the model in the history of science was based.

In contemporary literature on mathematical education research L. Radford (2010, 2013), following the ideas of K. Marx, adopts a position similar to that of V.V. Davydov: he writes that the eye can notice the essential features of the objects if it became the "theoretician" — the work of the receptive organ needs to be organized in a special cultural way.

How may a student acquire this special cultural, theoretical way of perception? In L. Radford's (2010) view, the transformation of the perceptive organ occurs during the spontaneous involvement of students into special cultural practice, established by the teacher in the class. Interacting with the teacher, the student not only perceives the meaning of the teacher's speech, but involves perceptive-motor, embodied interaction, which includes gestures, mimics, rhythm of the sentences, and intonations. Then "the senses... *become* shaped in certain historically formed ways as we engage in socio-cultural practices" (Ibid, p. 2). V.V. Davydov saw another source of transformation of perception in the course of education: he supposed that the specific actions for a given mathematical concept should emerge during the solving of the corresponding tasks; only after this would the students have acquired the necessary methods of action and be able to work with the visual model in the right way (Davydov, 1990/1972). Both researchers agree that the adequate perception is based on a reproduction, rooted in the historical, traditional method, of working with a visual model, acquired in the process of education.

Based on numerous experimental studies, A.V. Zaporozhets (2002/1986) summarized three stages of the perception development. The *first stage* is the stage of external material actions with the object; thus, for example, the student may trace the perpendicular with the pen or the index finger in order to find the coordinate. (Even pre-schoolers are able to invent such actions during independent experiments with the toy-puzzle — cultural object, covert inner relations of which are organized in accordance with the system of rectangular coordinates (Podd'iakov, 1992, 2001)). During the *second stage* the detailed process of perception is present, and perceptive actions "are performed with the aid of motions of receptor apparatuses and anticipate subsequent practical actions" (Zaporozhets, 2002/1986, p. 41). At this stage we expect to find the movement of the eyes along the path of the finger in the preceding stage. The *third stage* is the stage of the most formed perception, the stage of shortening and automation. But even the shortened automated action has genetic traces of the initial practical action: the rules of an algorithm for the ideal movement of attention across a field of perception correspond to the rules and limitations of those real actions, which were previously performed by the subject for the practical solution of an assigned task" (Ibid, p. 40). The last stage was studied in detail on the material of recognition by A.I. Podolsky, under the supervision of P.Ya. Galperin, in his work *Formation of simultaneous recognition* (Podolsky, 1978), in which he managed to simulate the path of perception development during the systematic gradual formation of the action in the task of the classification of visual material. Thus, the works of A.V. Zaporozhets and A.I. Podolsky show, firstly, the necessity of practical action in the process of the perception formation, and, secondly, the reduction of the orientating part of perceptive action up to simultaneous perception. It means on the one hand, that in ontogenesis we see "specifically human sensory education", during which "systems of sensory standards developed by society" (Zaporozhets, 2002/1986, p. 36) are acquired (we consider visual mathematical models as such standards), and, on the other hand, "reduction of informational processes and their merging with adaptive acts or executive actions" takes place (Zaporozhets et al., 1967, p. 30). Thus, we may expect that the experience of solving various tasks, including those that are not connected with our tasks, will be seen in the structure of perceptive actions of our subjects.

One of the reasonably obvious consequences of gaining experience is the shortening of the orientating part of actions, [which is] aimed towards detection of the perceptive field relevant to the task (for example, this is articulated in the hypothesis of "reduction of information" in the relatively recent work of H. Haider and P.A. Frensch, 1996): while solving particular tasks, people learn to select relevant information and ignore information, irrelevant to the task.

A brief historic-logical analysis of the elaboration of the Cartesian plane as a visual model of the "Cartesian coordinates" concept

The history of the development of the idea of the Cartesian plane goes back to antiquity, when the system of perpendicular lines was used independently in astronomy and geography for *visual* fixation of objects' locations. As early as Eratosthenes' *Geography* the author designated the longitudes and latitudes for 8000 points on a map (Burton, 2011, p. 184). The task presented in the empirical part of our study may de facto be already solved in the same way as finding coordinates of cities on a map, and the visual model of the "Cartesian coordinates" concept that we use corresponds to similar ways of finding coordinates of a point, or a point according to coordinates. But the obvious similarity of a mathematical model with the methods of recording objects' positions in other areas of knowledge just shows the convergence and joint evolution of the whole of cultural knowledge.

The development of the Cartesian coordinates concept in mathematics is connected with the task, which is essentially different from simple mapping and is based on the joining of *algebra and geometry* in one system. Attempts to set geometrically known curves (for example, the section of a cone) with algebraic functions and, vice versa, to depict algebraic relationships on a plane were found even in ancient mathematics (Boyer, 1944). But before the works of R. Descartes and P. Fermat[1] these attempts were local, and were made for particular curves. The crucial nature of the independent findings that were done by these two scientists is the proposition of the method of algebraic description of geometrical curves (Yushkevich, 1970) and, vice versa, depicting equations containing two variables: the segment of defined length was intercepted *along* the defined line, then the value of the second variable was calculated in terms of the defined value of the first variable and was intercepted *along* the second line (Boyer, 1944, p. 103).

Thus, the essence of the concept of Cartesian coordinates goes beyond the manipulation on a coordinate plane, which shows just one side of the concept. But the logic-historical analysis allows us to uncover the practical-object action that forms the basis of the development of visual model of this concept, which is the movement "*along*" the axes.

[1] To do the historical perspective justice, it is worth noting that before R. Descartes and P. Fermat the method of assigning curves through coordinates was represented in a rather obvious form in the works of N. Oresme (appr. 1361), but he didn't receive wide recognition (Boyer, 1944).

We may see the process of this development in the works of the pioneers: they didn't have a fixed visual model and the illustrated picture was drawn anew for each task: thus, in majority of cases only one axis was depicted and the direction of the second axis was signed. The axes were, as a rule, not perpendicular, and their directions varied from task to task (Yushkevich, 1970; Burton, 2011). Only gradually was the Cartesian plane with perpendicular axes established, and that brought the mathematical model closer to the practice of work with geographical and astronomic maps. It is clear that in such a model the *theoretical* action, which underlies model building and perception, is vertical and horizontal movement.

Another step toward the Cartesian plane of contemporary educational programs and our research was established only in the XVIII century: negative numbers were included in the model and directions were finally fixed for them in both axes (Yushkevich, 1970; Burton, 2011). For the usage of a Cartesian plane with the point (0, 0) in the middle of the picture, one has not only to understand the movement along the axes of coordinates, but also to choose in which direction to move in accordance with the coordinates' sign.

It should be stressed once more that the most important feature of the mathematical concept of "Cartesian coordinates" is the overlapping, in a single system, of algebra and geometry. We will see the importance of this fact when we find heuristics in the solutions of some tasks by experts; these heuristics are rooted in the wider contents of the "Cartesian coordinates" concept, rather than in the operations within the visual model.

Analysis of eye movements as a method for investigating the transformation of perception at different levels of competence

The analysis of eye-movements is a commonly used method in investigations of the perception of visual models and other visual symbolic means (Andrà et al., 2009, 2013; Carmichael, Larson, Gire, Loschky, & Rebello, 2010; Crisp, Inglis, Mason, & Watson, 2011; Epelboim & Suppes, 2001; Gegenfurtner, Lehtinen, & Säljö, 2011; Jarodzka, Scheiter, Gerjets, & van Gog, 2010; Moeller, Klein, Nuerk, & Willmes, 2013; Nyström & Ögren 2012; Peters, 2010; San Diego, Aczel, Hodgson, & Scanlon, 2006; Schneider, Maruyama, Dehaene, & Sigman, 2012; Susac, Bubic, Kaponja, Planinic, & Palmovic, 2014; Van Gog & Scheiter, 2010; Yang, Chang, Chien, Chien, & Tseng, 2013). Studies dedicated to the transformation of perception due to the increase in competence level have prominent place among these works. The confirmation of the hypothesis of "information reduction" (Haider & Frensch, 1996) is well established: experts are more able to select relevant areas of representations for the task, as they do more fixations in relevant areas than the novices do.

The authors of the hypothesis (Haider & Frensch, 1999) themselves conducted a series of experiments studying the nature of these phenomena. In one of the experiments the subjects were given rows of letters, for example "A B C D (4) I". The number in the brackets indicated the number of missed letters when listing them in alphabetical order and this number could be right or wrong (in this example there are 4 letters missing between D and I: E

F G H). The task for the subject was to verify if the number was correct. The number was put at the beginning or at the end of the row. Each subject had 8 blocks of 60 rows (30 right and 30 wrong). The eye-tracking method was used to find out at which stage the subjects started to ignore irrelevant information: at the stage of perception or at the stage of central information processing. The results of the experiment showed that those subjects in the process of training paid less and less attention to irrelevant information. Consequently, "information reduction" is already taking place at the stage of perceptive actions. In accordance with their findings the authors suggested two stages of the reduction process: at the first stage the differentiation between relevant and irrelevant information takes place; at the second stage, relevant information is actively perceived, whereas irrelevant information is ignored. By analyzing the performance of the task under the instruction to work without mistakes or with maximum speed, the conclusion was made that ignoring irrelevant information is under conscious control.

But for us other works are more interesting, such as those in which the perception of visual models as a result of additional knowledge acquisition in different areas of scientific knowledge was studied. For example, Canham and Hegarty (2010) studied eye-movements of geography students concerning weather maps. Students attempted to solve 30 tasks about the estimation of wind direction based on pressure maps; following this they completed training in the main principles of forecasting wind direction, and then they attempted the 30 tasks again. The dependent variables were the correctness of the answers and the time taken to complete the tasks, as well as the time of fixations in relevant and irrelevant areas of interest (AOI). The results from the tasks both before and after the training were compared. After the training the percentage of correct answers increased, but so too did the time taken to complete the tasks. Subjects were spending significantly more time looking at the relevant areas of interest and less time at the irrelevant areas (such as the map's temperature legend). Thus, even brief training may influence perception: we start to ignore irrelevant information and pay more attention to relevant information.

In many other studies the participants were not trained during the experiment, but the perception of subjects of different levels of competence – novices and experts – was compared. For areas of knowledge such as medicine, transport, sport (Gegenfurtner, 2012), zoology (Jarodzka et al., 2010), physics (Carmichael et al., 2010), paleontology (Yang et al., 2013) and so forth there is evidence that experts are better at selecting relevant information in visual pictures or texts, that their gaze dwells in relevant areas for a longer time and that more fixations are observed, while novices pay attention mostly to perceptively bright details.

But only a few works are dedicated to the study of perception of mathematical material, and fewer still are dedicated to the perception of visual models by participants with different levels of mathematical competence. Case study analysis showed that during the solving of geometrical tasks experts focused on the relevant area, where there were no pictures but additional lines needed be drawn in order to solve the task (Epelboim & Suppes, 2001). Peters (2010) in his case study compared the reading of the texts of mathematical tasks and found out that the expert mathematician chose more mathematically significant parts of the text and she was able to capture their meaning by shorter fixations, while for a novice it was necessary to dwell on the task for a long time in order to interpret the value of numbers and variables. Crisp et al. (2011) studied how students figured out functional relationships between the variables from the value table. It was supposed that the proportion of vertical and horizontal saccades may differ between novices and experts, but results showed that differences mostly depended on individual strategy and not on the level of competence.

Empirical research

The aim of our empirical research was to explore the dependence of perceptive actions in solving tasks on the visual model of Cartesian coordinates according to the level of mathematical competence. The question for the research was whether it is true that "competent" perception of this visual model includes specific "theoretical" actions that have been revealed during our historical analysis.

Let's consider the choice of parameters for analysis. Most of the studies in the area of psychology of mathematics education that have used eye-tracking are from the position of semiotic paradigm (Duval, 2006, 2008; Hitt, 1998), where a mathematical concept is considered as an association of several sign representations of various modalities: text, formula, pictures. There were studies of the number of saccades between the representations, the length and number of fixations in the area of one representation, namely the transition sequence between different representations of the same material. Andrà et al. (2013) arbitrarily distinguished three levels of analysis of representations perception: *macro-level* (analysis of frequency and sequence of gaze attendance of each representation, for example Andrà et al., 2009, 2013; Nyström & Ögren, 2012), *medium level* (analysis of eye-movements within one representation, for example, Kuravsky et al., 2013; Peters, 2010; Susac et al., 2014; Crisp et al., 2011), and *micro-level* (analysis of attendance of a particular part of the representation, such as a concrete number (Schneider et al., 2012; Moeller et al., 2013) or a particular area (Epelboim & Suppes, 2001)). In some rare research studies various levels are combined in order to describe how various representations are intertwined into a unified understanding during actual mathematical activity (e.g. San Diego et al., 2006).

In our research we follow ideas developed by V.V. Davydov and consider a mathematical concept as being based on actions. From the perspective of the activity theory of thinking, all data about "attraction" of experts' gaze to relevant areas of representations may be explained as reorganization of the perception process according to acquired theoretical knowledge. The various aspects of theoretical knowledge in culture (in particular our own culture, within a framework of which the study is conducted) are regarded as products of the shortening of the methods of action, which occurred as an answer to particular tasks that are rooted in the process of the science's historical development.

We are interested in the operational characteristics of eye-movements concerning a visual model (one of representations), and our analysis is the combination of the medium-level analysis (specifics of saccades within the visual model) and the micro-level analysis (attendance of particular significant zones). We hypothesized that the directions of saccades reflected the specific method of work with the Cartesian plane that has developed in culture: movements along the axes. Consequently, we expected that vertical and horizontal saccades would prevail over saccades along other directions.

Another piece of theoretical knowledge that could transform the process of perception is the correspondence between the sign of the coordinate and the direction of the axis. We supposed that the ability of experts to use this knowledge would lead to instantaneous detection of the target quadrant of the coordinate plane as the most relevant one and, consequently, to the greater number of fixations in the relevant quadrant by experts.

Subjects. Subjects from three levels of competence took part in the study. In the expert group there were 11 subjects, all of whom had graduated from mathematics departments; in the middle group there were 23 first-grade students of non-mathematical specialties (they all had passed the final exam in mathematics at school); in the weak group there were 10 school students of the 9–11th grades, who were still studying Cartesian coordinates at school.

Equipment. We used the SMI RED eye-tracker with a frequency for registering the location of the gaze set at 120 Hz. The recording was done using the IviewX program, stimuli were presented by Experiment Center 3.1, and Begaze 3.1 and SPSS 20.0 were used for the data analysis. Subjects were seated 40–50 cm from the monitor. A test series was preceded by a 9-point calibration with validation; subjects were accepted for participation in the full study only if they achieved a calibration accuracy of .5 grad.

Procedure and materials. At the beginning of the experiment the following instructions were given: "Now you will see the tasks on a Cartesian coordinate system. Try to solve them as quickly and as accurately as possible". Then each subject attempted to solve 10 tasks involving a visual search for a point on a Cartesian plane with the given coordinates. The instruction for the each of the tasks was as following: "Choose a point with coordinates (3, −4)" After reading the task each subject saw a Cartesian plane and 4 points, A, B, C, D, on the screen. There were either 1 or 2 points in the target quadrant of the plane. The subject had to memorize which point had the given coordinates, and on the next screen choose the right answer using a mouse. Thus, each task had 3 slides: a task, the Cartesian plane, and the answer screen. There was no time limit for solving the tasks and participants switched to the next slide by pressing the "space" key. One of the tasks was "provocative" and had no correct answer, but we excluded the data from that task from the analysis of the present paper.

Hypotheses. Initially we formulated the following hypotheses:

1. (A) Saccades of horizontal and vertical directions prevail over the saccades

along other directions. (B) This proportion is more overt for the more mathematically competent subjects.

2. More competent subjects make fewer fixations in irrelevant quadrants of the plane.

3. Perceptive actions are shortening progressively with better acquisition of mathematical knowledge: more competent subjects have a shorter gaze path in the task-solving, fewer fixations and a take less time to solve the tasks.

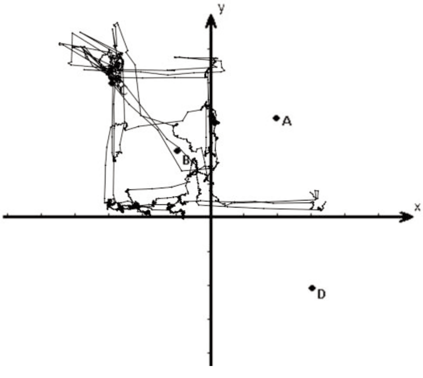
Processing of data and results

The first part of our analysis is devoted to the directions of saccades. The standard algorithm of saccade detection in Begaze 3.0 identifies a saccade as a vector between the centers of the fixation before the saccade and the fixation after the saccade. But careful study of the raw data (Figure 1 as an example) showed that the directions of many saccades were distorted by this method because of the considerable drift during the fixation towards one of the sides (then the beginning of the fixation was calculated not from the point from which the eye started to move with high speed, but rather from a far point — the middle of the drift). We developed additional software for the identification of saccades based on the simple algorithm of threshold velocity of eye movements (Salvucci & Goldberg, 2000). We considered the movement of the eye as a saccade with a velocity higher than 120°/sec; this allowed us to calculate the direction of the saccade as starting at the point where speed exceeded the threshold, and finishing at the point where speed decreased below the threshold. The direction of a saccade was calculated as an angle from 0° to 90°: the saccades that were close to vertical were marked as 90°, and saccades that were close to horizontal were marked as 0°. All saccades were divided into 6 sectors of 15 degrees according to their direction: from sector 0°–15° till sector 75°–90°. Saccades of the first and the last sectors were considered as horizontal and vertical correspondingly.

The mean number of the saccades in each sector was compared with the

Figure 1

The example of the raw data of the eye movements during the search of the point with coordinates (−3; 4)



ANOVA with repeated measurements; the level of mathematical competence was defined as the between-group factor. Saccades of vertical and horizontal sectors were observed approximately 4 times more often than saccades of other sectors ($F = 31.554$, $p <. 001$, Figure 2).

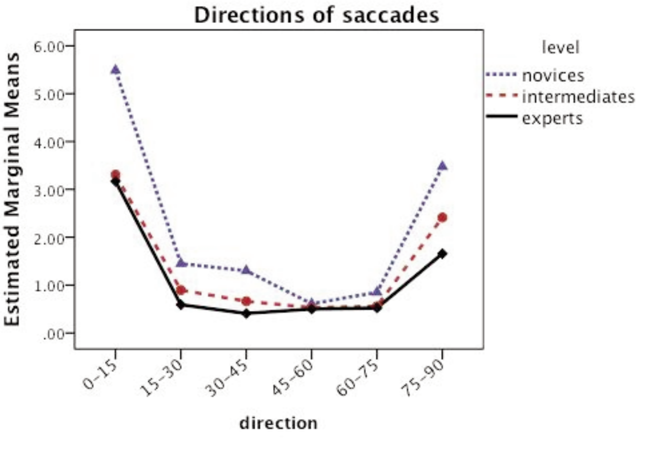
This relationship was rather stable between the groups. However, the number of saccades dropped significantly with an increase in mathematical competence ($F = 5.446$, $p = .008$). Thus, we may say that the first hypothesis was partly confirmed: the prevalence of vertical and horizontal saccades over the saccades along other directions was seen for all the groups, but we failed to show that an increase in competence is accompanied by an increase in the prevalence of vertical and horizontal saccades. The third hypothesis, about the shortening of perceptive actions with the increase in competence, was confirmed.

We've seen significant interaction of factors ($F = 3.395$, $p = .043$, taking into account the most strong Lower-bound correction for the absence of sphericity). To find the sources of this interaction we calculated the relative quantity of the saccades of various directions for each subject (this allowed us to exclude the factor of general shortening of perceptual actions with the increase in competence). We matched the relative quantity of vertical and horizontal saccades in different groups and again found an interaction between the factor of the saccades' direction (2 levels: vertical or horizontal) and the group factor of mathematics competence (F=4.218, p=.022). It turned out that expert mathematicians made significantly more horizontal saccades than vertical (47% and 23.7% correspondingly); this difference was less observable for school students (41.6% и 28%), and smaller still for university students (38% и 30%).

Following the verification of the third hypothesis, we compared 1) the number of fixations, 2) the length of the

Figure 2

Mean number of the saccades of various directions per one task and three groups of subjects



gaze path and 3) the general time taken for task-solving by ANOVA with repeated measures (each of three variables was analyzed separately). The tasks were considered as an intraindividual factor and the level of competence was considered as a between-group factor. Mauchly's Test of Sphericity showed significant difference from the spherical model ($p < .001$), so we used the Lower-bound correction for the estimation of factors interaction.

Analysis showed significant differences between the groups for all quantitative parameters ($p < .05$, see Table 1 for the more detailed statistics), and the significant influence of the factor of the task ($p < .001$). The interaction of factors of tasks and the group was also revealed for each of the parameters. In order to find out the source of the interaction we'll observe in more detail the variability of the mean number of fixations, as seen in Figure 3.

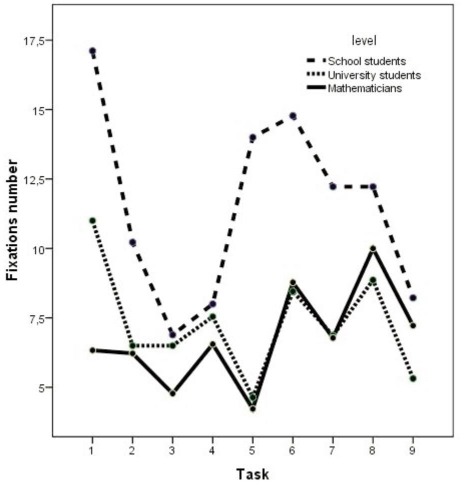
Table 1

Parameters that showed the shortening of perceptive actions with the increase in competence

Parameters	**Level of competence**			**ANOVA results**	
	School students	University students	Mathematicians	*F*	*p*
Time of task-solving [sec]	4.638	3.285	2.681	4.916	.013
Quantity of fixations	14.02	9.8	7.54	5.794	.006
Length of path [pixels]	1810.2	1250.1	814.5	5.744	.007

Figure 3

Mean number of fixations during the task-solving in different groups



A marked increase in the number of fixations by the school students was observed in tasks 5, 6, 7, and 8. All these tasks, as well as task 2, had 2 points in the target quadrant: the point with the given coordinates and the distracter-point. In Figure 3 it is seen that the distracter-point in the target quadrant had little influence on the number of fixations by the university students and the mathematicians, though it significantly lengthened the process of visual orientation for the school students. In order to verify this suggestion by statistics we compared the mean number of fixations for each group in the tasks with 2 points in the target quadrant with those in the task with 1 point (see Table 2) (we excluded the first task from the analysis). We observed significant interaction between the factor of the number of points in the target quadrant and the factor of competence level ($F = 8.249$, $p = .001$). The presence of a distracter in the target quadrant increased the number of fixations only in the group of school students, but had no significant influence in the other groups (see Table 2).

Moreover, Figure 3 shows that the numbers of fixations by the school and of university students dropped significantly from the first task to the next, reflecting the shortening of the perception processes not only from the less competent subjects to the more competent ones, but also during the experiment: we compared the number of fixations during the first and the third task (both tasks have one point in target quadrant). Significant differences were found for the group of school students ($t = 2.318$, $p = .042$) and for the group of university students ($t = 2.547$, $p = .014$), while for mathematicians no significant difference was observed.

On the whole, these results confirm the third hypothesis about the shortening of the perception process from less competent subjects to the more competent, as well as from the first task to the next.

The next aim of our analysis was to study the ability of subjects from different groups to use the signs of coordinates for orientation. We divided the whole coordinate plane into 6 areas of interest: 4 quadrants and 2 coordinate axes (see Figure 4). It is possible to figure out to which quadrant the target point belongs, taking into account only the signs of the coordinates, so we considered the other 3 quadrants as irrelevant for the task. Since absolute values of the general number of fixations differed between the groups, for this analysis we compared the percentage of fixations in irrelevant areas by subjects from different groups. The significant influence of the factor of competence level on the frequency of fixations in irrelevant areas was found. Post-hoc

Table 2

Mean numbers of fixations in tasks with one or two points in the target quadrant

	One point in the target quadrant	Two points in the target quadrant
School students	11.11	14.55
University students	9.44	8.35
Mathematicians	7.59	7.58

Figure 4

Six areas of interest: 2 axes and 4 quadrants

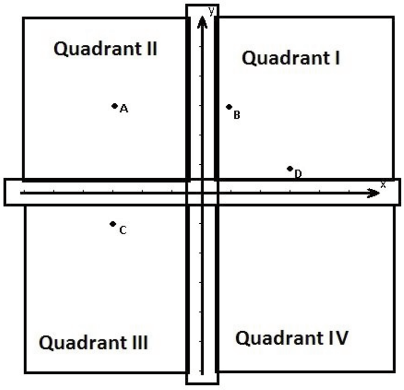
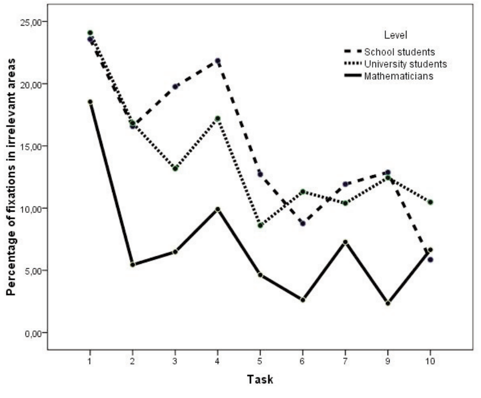


Figure 5

The percentages of the fixations in the irrelevant areas



analysis (using the Scheffe Test) showed significant differences between the mathematicians and university students (p = .007), and between mathematicians and school students (p = .01), whereas the difference between the school students and the university students was not significant (see Figure 5).

Let us turn to the qualitative analysis which we conducted with a hope of revealing strategies in task solving that would reflect the involvement of knowledge of the Cartesian coordinates concept, which was richer than the visual model. Watching the individual pathways of eye movements, we noticed that the vertical-horizontal pattern of the gaze path, which was typical for all subjects, was used very rarely by experts in some tasks. In particular, in task 6 it was needed to find a point with coordinates (−4; −4). Obviously, this point lies on a line, starting from the beginning of the coordinates under 45°

in the third quadrant. In this task there was a distracter-point with coordinates (−2; −2). Some subjects found the target point without counting its coordinates on the axes, as they did in other tasks, but they moved along the diagonal with a transitional fixation on the point (−2; −2) and only afterwards did they check the coordinates of the target point (−4;−4) (see Figure 6). We analyzed how many participants in each group fixated the point (−2; −2) and how many participants made diagonal saccades from (−2; −2) to the point (−4; −4). The analysis of frequencies (see Table 3 for raw data) using the crosstabs statistic showed that fixations on the point (−2; −2) occurred significantly more often ($\chi^2 = 7.212$, $p = .028$) in the group of experts than in the groups with lesser mathematical competence. The difference in the frequency of diagonal saccades was found at the level of tendency ($p = .09$); such saccades were performed by 7 experts out of 11, and they appeared less often in the other groups (3 subjects out of 10 for the school students and 6 out of 23 for the university students).

Discussion

As was hypothesized, we found specific perceptive actions on which the

Figure 6

Search of the point (−4, −4) in the group of experts

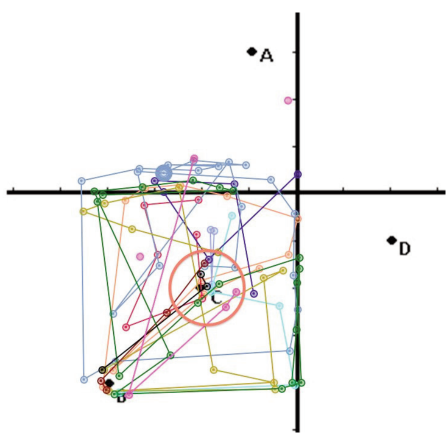


Table 3

Numbers of fixations on the distracter-point (-2; -2) that was on the way to the target point

Fixations on the point (−2, −2)	School students	University students	Mathematicians	All
No	6	12	1	19
Yes	4	11	10	25
All	10	23	11	44

work with the visual model of Cartesian coordinates is based: vertical and horizontal saccades along the axes of coordinates were much more common than the saccades along other directions. This distribution of the saccades' directions was typical for all three groups of subjects independently from their level of mathematical competence. So we did not reveal clear evidence that the formation of a specific cultural method of perception that includes "theoretical" actions with a visual model (Davydov, 1990/1972) took place during the education process (as Zaporozhets, 2002/1986 and Radford, 2010, 2013 have assumed).

It is possible that these specific perceptive actions were already formed in the 9–11th grades at school, and so our group of school students was not "naïve" enough, If so, this may have prevented us finding the stage in perception development when this pattern doesn't seem predominant in solving these tasks. In order to check this interpretation we need to gather data about such task-solving by even less competent subjects: schools students who are just starting to study Cartesian coordinates or have not even seen this visual model before. If we manage to find another distribution of saccades' directions in the perception of junior school students, we will be able to confirm that this pattern of eye movements is the result of the acquisition of specific "theoretical" actions of perception.

However, another interpretation is also possible. It is known that the human visual system has greater sensibility and accuracy in regard of vertical and horizontal directions (see, for example, Campbell, Kulikowski, & Levinson, 1966). One possible interpretation of this is the large number of verticals and, especially, horizontals in the human environment (Dragoi, Turcu & Sur, 2001; Coppola, Purves, McCoy & Purves, 1998). Correspondingly, it is possible that vertical and horizontal saccades in eye movements reflect not only perceptive actions specific for the Cartesian system of coordinates, but also eye movements along the most ecologically significant directions that coincide with the axes of coordinates. This hypothesis may be tested by presenting subjects with a rotated Cartesian coordinates system, such that movements along the axes of coordinates won't be orientated vertically or horizontally.

In any case, vertical and horizontal saccades are present already in relatively early stages of perception development during the learning of the Cartesian coordinates system. As we have shown, with an increase in competence level, a shortening of perception processes occurred; it corresponds to our third hypothesis: experts solve the task on visual search for the point with the given coordinates quicker, with fewer saccades and fixations and passing a lesser gaze path (see Table 1). This difference was especially strong in the solving of the first task (see Figure 3): the school students needed 3 times more, and the university students approximately 1.5 more, fixations for its solving than subjects with higher mathematical education needed. In the third task this difference almost vanished and remained only for the tasks with the distracter-point in the relevant quadrant. Interestingly, the distracter in the relevant quadrant of the coordinate plane extended the solving-time only in the case of school students, while the university students, who had already repeated all the school material and were some way beyond the level of the school program, were not disoriented by this distracter. It can be said that the appearance of the distracter in the relevant quadrant destroys the automated process of perception and returns the school students to the previous stage of perception development, which consist not only in executive perceptive actions but also in detailed orientation activity (Zaporozhets, 2002/1986).

Interestingly, the expert mathematicians made more horizontal saccades than vertical ones in comparison with the analogous proportions for the university students and the school students. Indeed, in order to choose a point with the given coordinates from the suggested four points, it is usually enough to analyze just one coordinate: the abscissa in the symbolical reference of the point is always in the first place. The exceptions are the cases when one of the coordinates of the distracter-point coincides with the coordinate of the target point. We analyzed our stimuli and came to the conclusion that this moment was not controlled: in some tasks there were points positioned on the same horizontal, but not on the same vertical. Thus, horizontal saccades, rather than vertical ones, were needed for solving the majority of the tasks. Their predominance in the experts' eye-movement activity is further evidence that experts perform only necessary executive perceptive actions, while the orientation part of perception is reduced in their eye-movement behavior.

In the research of Crisp et al. (2011), which was conducted on the material of tables' perception, the ratio of the vertical and horizontal eye movements was also explored; it was found that relative quantity of vertical and horizontal saccades reflect individual strategy, which one individual uses in different tasks, and does not depend on the competence level of the subjects. Probably, more detailed analysis of separate tasks in our case may also reveal individual strategies for the tasks involving Cartesian coordinates, but it goes beyond the scope of this article.

Let's return to the analysis of the differences between the students and the experts: the shortening is not the only difference in the processes of perception between the subjects of various groups. Our second hypothesis, about the smaller number of fixations in irrelevant areas, was also confirmed: it is the mathematicians (as distinct from the university students and the school students) who used far fewer fixations in irrelevant quadrants of the coordinate plane: just 7.1% fixations for the mathematicians against 13.8% and 14.9% for the other groups. These results correspond to numerous data showing that experts pick out significant parts of visual representations (Canham & Hegarty, 2010; Gegenfurtner, 2012; Jarodzka et al., 2010). Also these results correspond to the data of Andrà et al. (2009) that novices more often examine alternative answers, while the experts' choices are based on analysis of the task's description. We assume that it is at the expert-knowledge stage when information about the signs of the coordinates is taken into account during the programming of perceptive actions: the expert mathematicians immediately discarded three of the four quadrants as irrelevant.

The understanding that the point with the similar coordinates lies on the diagonal was revealed as another piece of additional knowledge that was used primarily by experts for the organization of their process of perception. This knowledge goes beyond the scheme of the action that allows finding longitudes and latitudes of cities on a map and ascends to a more general understanding of the concept of Cartesian coordinates, in particular the coordinates of points that lie on the diagonal line. The data show that it was the experts, as opposed to the university students and the school students, who detected the point with coordinates (−4, −4) by tracking the direction under 45°, which lead directly to the target point. They fixated on the point (−2; −2), which lay on this diagonal, while in other tasks the experts usually did not fixate on distracter-points; so the additional knowledge was involved in a particular situation and had restructured the process of perception. We assume that by choosing the particular tasks it's possible to reveal an array of heuristics that competent mathematicians would use while solving these tasks; we suppose that these heuristics would be the outcome of deeper theoretical understanding of the concept of Cartesian coordinates. The aspects of the concept, which go beyond the visual model of a Cartesian plane, were acquired by the subjects in the course of solving a wider scope of the tasks than just matching the points and the coordinates; by solving other tasks, other perceptual actions were shortened into *theoretical* perception.

On the whole we have shown that the perception of school students in comparison with the perception of university students has a more detailed algorithm of task-solving which is made more vulnerable by distracter-points in the target quadrant; the school students' and the university students' perceptions reflect low and high stages of development and weak and strong stability of the algorithmic method of task-solving: "Check one coordinate — check the other coordinate". The perception of expert mathematicians, as opposed to that of the university students and the school students, is characterized by flexible employment of additional knowledge (instead of using only one method of action that characterized the "initial form" of the concept, as V.V. Davydov supposed). We may agree with the followers of J. Piaget that well-acquired mathematical knowledge functions as a synchronized system of various schemes of action (Dubinsky & McDonald, 2001) and that it is organized into "conceptual fields" (Vergnaud, 2009) — sets of situations where different concepts-in-action function and become flexibly involved in accordance with the needs of particular tasks.

Conclusions

According to the findings, the perception of a visual model formed during education is organized according to the actions that were established during the elaboration of the visual model in the history of mathematics: the eye movements of students and expert mathematicians correspond to the movements along the axes of the Cartesian plane during the search for a point with given coordinates. With an increase in mathematical competence, perceptive actions shorten and the task is solved more quickly and with fewer fixations. But besides the general decrease of necessary orientation in the task, specific changes occur. The difference between school students, who are still learning the material, and university students, who have already passed their math exam, is the stronger stability of the algorithm in solving the perceptive task to the distracter-point that is exposed close to the target point. Expert mathematicians differ from the students by the transformation of perceptive processes under the influence of additional knowledge, in particular those that are adequate to the concrete situation: experts more often use knowledge about the correspondence between the signs of the coordinates and the directions of the axes, or knowledge about location of the points with equal coordinates on the diagonal.

In the future, more detailed analysis of the emergence of vertical and horizontal movements along the axes of coordinates in ontogenesis is needed in order to separate the cultural factor of education from the natural human preference of these directions as basic for the environment. However, the available data allow us to distinguish the *theoretical* perception of a visual mathematical model that was formed during education: namely the perception, which includes the specific methods of interaction with the model, which becomes more and more laconic, more stable and more simple with an increase of mathematical competence, and which enriches the set of possible operations due to the integration of all mathematical knowledge into the process of perception.

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Перцептивные действия у учащихся и экспертов при использовании визуальной математической модели

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Резюме

Исследование направлено на изучение перцептивных действий, позволяющих воспринять изображение как репрезентирующее математическое понятие. Работа основана на культурно-историческом подходе, развиваемом В.В. Давыдовым в отношении теоретического и, в частности, математического мышления, в котором математическое понятие полагается отражающим исторически обусловленный способ действия. В исследовании анализируются различия в процессах восприятия визуальных моделей учащимися школьного и студенческого уровня и экспертами с высшим математическим образованием. В ходе анализа глазодвигательной активности при решении задачи на зрительный поиск точки на декартовой плоскости используется традиционный для исследований восприятия экспертов и новичков анализ длительности посещения релевантных и не релевантных зон интереса, сопоставляются такие количественные показатели решения задач испытуемыми разных групп, как длина пути взгляда, общее время решения задачи, количество фиксаций. Кроме того, анализируются направления саккад, с целью выявить движения взгляда вдоль осей координат, свидетельствующие об исторически обусловленном способе действия при работе с понятием декартовых координат. Также исследуется использование специфических эвристик, применяемых экспертами для решения некоторых задач.

Согласно нашим данным, при работе с декартовой плоскостью действительно преобладают вертикальные и горизонтальные саккады, направленные вдоль осей. Кроме того, с ростом математической компетентности происходит, с одной стороны, сворачивание ориентировочной составляющей восприятия, с другой стороны, гибкое привлечение дополнительных математических знаний уже на уровне построения перцептивных действий. Это говорит о необходимости учитывать в конкретной практике математического образования, что учащиеся воспринимают визуальную модель принципиально иначе, чем это делают их преподаватели-эксперты, и что кажущаяся наглядность может оборачиваться непониманием вследствие невладения специфическими способами восприятия. Общепсихологическим выводом исследования является принципиальное сплетение понятийных структур и процессов зрительного восприятия, организуемого сообразно целостной системе знания. Особенности восприятия декартовой плоскости экспертами соответствуют более поздним этапам исторического развития этой визуальной модели, что эмпирически подтверждает правомерность использования термина «теоретическое восприятие».

Ключевые слова: логико-исторический анализ, восприятие, визуальная модель, математическое понятие, перцептивные действия, декартовы координаты, запись движений глаз, новички и эксперты, психология математического образования.